Strong and weak clustering of inertial particles in turbulent flows

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We suggested a theory of clustering of inertial particles advected by a turbulent velocity field caused by an instability of their spatial distribution. The reason of the *clustering instability* is a combined effect of the particle inertia and finite correlation time of the velocity field. The crucial parameter for the instability is a size of the particles. The critical size is estimated for a *strong clustering* (with a finite fraction of particles in clusters) associated with the growth of the mean absolute value of the particles number density and for a *weak clustering* associated with the growth of the second and higher moments. A nonlinear mechanism for a saturation of the clustering instability (particles collisions in the clusters) is suggested. Applications of the analyzed effects to the dynamics of aerosols and droplets in the turbulent atmosphere are discussed. The critical size of atmospheric aerosols and droplets in clustering is of the order of $(20-30)\mu$m, and a lower estimate of the number of particles in a cluster is about hundreds.

I. INTRODUCTION

Formation and evolution of aerosols and droplets inhomogeneities (clusters) are of fundamental significance in many areas of environmental sciences, physics of the atmosphere and meteorology (*e.g.* smog and fog formation, rain formation), transport and mixing in industrial turbulent flows (like spray drying, pulverized-coal-fired furnaces, cyclone dust separation, abrasive water-jet cutting) and in turbulent combustion, see *e.g.* [1] and references therein. Analysis of experimental data shows that spatial distributions of droplets in clouds are strongly inhomogeneous [2]. Small-scale inhomogeneities in particles distribution were observed also in laboratory turbulent flows [3,4].

It is well-known that turbulence results in a relaxation of inhomogeneities of concentration due to turbulent diffusion [1], whereas the opposite process, *e.g.*, a preferential concentration (*clustering*) of droplets and particles in turbulent fluid flow still remains poorly understood.

In this Letter we suggest a theory of clustering of particles and droplets in turbulent flows. The clusters of particles are formed due to an instability of their spatial distribution suggested in ref. [5] and caused by a combined effect of a particle inertia and a finite velocity correlation time. Particles inside turbulent eddies are carried out to the boundary regions between them by inertial forces. This mechanism of the preferential concentration acts in all scales of turbulence, increasing toward small scales. An opposite process, a relaxation of clusters is caused by a scale-dependent turbulent diffusion. The turbulent diffusion decreases towards to smaller scales. Therefore, the clustering instability dominates in the Kolmogorov inner scale η, which separates inertial and viscous scales. Exponential growth of the number of particles in the clusters is saturated by their collisions.

II. QUALITATIVE ANALYSIS: INSTABILITY GROWTH RATES FOR STRONG AND WEAK CLUSTERING

For an estimate of cluster growth rates we use the equation for the number density $n(t,\boldsymbol{r})$ of particles advected by a turbulent velocity field $\boldsymbol{u}(t,\boldsymbol{r})$:

$$\partial n/\partial t + \boldsymbol{\nabla}\cdot(n\boldsymbol{v}) = D\Delta n\,, \qquad (1)$$

where D is the coefficient of molecular (Brownian) diffusion. Due to inertia of particles their velocity $\boldsymbol{v}(t,\boldsymbol{r}) \neq \boldsymbol{u}(t,\boldsymbol{r})$. Equation (1) conserves the total number of particles. Equation for $\Theta(t,\boldsymbol{r}) = n(t,\boldsymbol{r}) - \bar{n}$ follows from (1):

$$\partial\Theta/\partial t + (\boldsymbol{v}\cdot\boldsymbol{\nabla})\Theta = -\Theta\,\mathrm{div}\,\boldsymbol{v} + D\Delta\Theta\,, \qquad (2)$$

where $\bar{n}$ is the uniform mean number density of particles, and the mean particles velocity is zero. We neglected the term $\propto \bar{n}\,\mathrm{div}\,\boldsymbol{v}$ describing an effect of an external source of fluctuations which is usually weaker than the effect of self-excitation of fluctuations. In order to elucidate particles clustering we analyze a role of different terms in eq. (2) using a reference frame moving with a cluster velocity $\boldsymbol{V}_{\rm cl}(t)$. The advective term, $[(\boldsymbol{v}-\boldsymbol{V}_{\rm cl})\cdot\boldsymbol{\nabla}]\tilde{\Theta}$, causes turbulent diffusion inside the cluster with the coefficient $D_{\rm T} \sim \ell_{\rm cl} v_{\rm cl}/3$, where $\ell_{\rm cl}$ is the characteristic size of a cluster and $v_{\rm cl}$ is the turbulent velocity at the scale $\ell_{\rm cl}$ and $\tilde{\Theta}(t,\boldsymbol{r})$ are the fluctuations of the number density in the comoving frame. The term $\propto -\tilde{\Theta}\,\mathrm{div}\,(\boldsymbol{v}-\boldsymbol{V}_{\rm cl})$ can result in an instability. Indeed, neglecting diffusion we find the solution of eq. (2): $\tilde{\Theta}(t,\boldsymbol{r}) \sim \tilde{\Theta}(0,\boldsymbol{r})\exp[-I(t)]$. Here

$$I(t) \equiv \int_0^t b(\tau)\,d\tau = \sum_{n=1}^{N} I_n\,,\ \ I_n \equiv \int_{n\tau_v}^{(n+1)\tau_v} b(\tau)\,d\tau\,,$$

and $b(\tau) = \mathrm{div}[\boldsymbol{v}(\tau, \boldsymbol{r}) - \boldsymbol{V}_{\rm cl}(\tau)]$ and $N = t/\tau_v \gg 1$. We neglected a spatial dependence of $b(\tau, \boldsymbol{r})$ inside the cluster and consider $b(\tau)$ as a random process with a correlation time τ_v which is of the order of turnover time of $\ell_{\rm cl}$-scale eddies, $\tau_v \sim \ell_{\rm cl}/v_{\rm cl}$. Now the integrals I_n can be considered as independent random values and the sum $I(t)$ is estimated by the central limit theorem: $I(t) \sim \sqrt{\langle \tau_v b^2 \rangle_v \, t}\, \zeta$, where $\langle \ldots \rangle_v$ is averaging over turbulent velocity ensemble, ζ is a Gaussian random variable with zero mean and unit variance. Now, averaging over statistics of ζ by the identity $\langle \exp(k\zeta) \rangle_\zeta = \exp(k^2/2)$ (with $k = \sqrt{\langle \tau_v b^2 \rangle_v \, t}$), we estimate $\langle |\Theta|^q \rangle \sim \langle |\Theta_0|^q \rangle \exp(\gamma_q t)$ with the growth rate γ_q of the q-th moment given by

$$\gamma_q \sim \tfrac{1}{2} \langle \tau_v (\mathrm{div}\, \boldsymbol{v})^2 \rangle_v q^2 - D_{\rm T}\, q \,/ \ell_{\rm cl}^2 \; . \tag{3}$$

Here the turbulent diffusion inside the cluster is crudely taken into account by the the term $\propto D_{\rm T}$. Clearly, the instability is caused by a nonzero value of $\langle \tau_v (\mathrm{div}\, \boldsymbol{v})^2 \rangle$, *i.e.,* by a *compressibility of the particle velocity field* $\boldsymbol{v}(t, \boldsymbol{r})$.

Compressibility of fluid velocity itself $\boldsymbol{u}(t, \boldsymbol{r})$ (including atmospheric turbulence) is often negligible and $\mathrm{div}\, \boldsymbol{u} = 0$. However, due to the effect of particle inertia their velocity $\boldsymbol{v}(t, \boldsymbol{r})$ does not coincide with $\boldsymbol{u}(t, \boldsymbol{r})$ [6], and a degree of compressibility of the field $\boldsymbol{v}(t, \boldsymbol{r})$, σ_v, defined by

$$\sigma_v \equiv \langle [\mathrm{div}\, \boldsymbol{v}]^2 \rangle_v / \langle |\boldsymbol{\nabla} \times \boldsymbol{v}|^2 \rangle_v \; , \tag{4}$$

may be of the order of 1 [5,7,8]. For inertial particles $\mathrm{div}\, \boldsymbol{v} \sim \tau_{\rm p} \Delta P / \rho$, where $\tau_{\rm p}$ is the Stokes time, $\tau_{\rm p} = m_{\rm p}/6\pi\rho\nu a = 2\rho_{\rm p} a^2/9\rho\nu$, $m_{\rm p}$, $\rho_{\rm p}$ and a are the mass, density and radius of particles, respectively. The fluid flow parameters are: the viscosity ν, density ρ, pressure P, Reynolds number $\mathcal{R}e = L u_{\rm T}/\nu$ and the dissipative scale of turbulence $\eta = L\, \mathcal{R}e^{-3/4}$. Now we can evaluate:

$$\sigma_v \simeq (\rho_{\rm p}/\rho)^2 (a/\eta)^4 \equiv (a/a_*)^4 \tag{5}$$

(see [8]), where a_* is a characteristic radius of particles for which $\sigma_v = 1$. For water droplets in the atmosphere $\rho_{\rm p}/\rho \simeq 10^3$ and $a_* \simeq \eta/30$. For the typical value of $\eta \simeq 1$mm this yields $a_* \simeq 30\mu$m. On windy days when η decreases, the value of a_* correspondingly becomes smaller.

Then we estimate $\langle \tau_v [\mathrm{div}\, \boldsymbol{v}]^2 \rangle_v$ as $2\sigma_v/\tau_v$. Taking the cluster size $\ell_{\rm cl}$ of the order of the inner scale of turbulence, η, we have to identify τ_v with a turnover time of eddies in the inner scale η, $\tau_v \to \tau_\eta \equiv \eta/v_\eta \simeq (L/u_{\rm T})\, \mathcal{R}e^{-1/2}$. Thus, the growth rate γ_q may be evaluated as

$$\gamma_q \simeq \gamma_{\rm cl}\, q (q - q_{\rm cr}) \, , \quad \gamma_{\rm cl} \sim \sigma_v / \tau_\eta \, , \quad q_{\rm cr} \sim 1/3\sigma_v \; . \tag{6}$$

It is clear that moments with $q > q_{\rm cr}$ are unstable. It follows from eqs. (5) and (6) that this happens when $a > a_{q,\rm cr}$ where $a_{q,\rm cr} = a_{1,\rm cr}/q^{1/4}$ is the value of a at which $q_{\rm cr} = q$. The largest value of $a_{q,\rm cr}$ corresponds to the instability of the first moment, $\langle |\Theta| \rangle$: $a_{1,\rm cr} \sim 0.8\, a_*$, $a_{2,\rm cr} \approx 0.84\, a_{1,\rm cr}$, $a_{3,\rm cr} \approx 0.76\, a_{1,\rm cr}$, $a_{4\rm cr} \approx 0.71\, a_{1,\rm cr}$, *etc.*

Note that if $\langle |\Theta| \rangle$ grows in time then almost all particles can be accumulated inside the clusters (if we neglect a nonlinear saturation of such growth). We define this case as a *strong clustering.* On the other hand, if $q_{\rm cr} > 1$ the first moment $\langle |\Theta| \rangle$ does not grow and the clusters contain a small fraction of the total number of particles. This does not mean that the instability of higher moments is not important. Thus, *e.g.*, probability of binary particles collisions is proportional to the square of their density $\langle n^2 \rangle = (\bar{n})^2 + \langle |\Theta|^2 \rangle$. Therefore the growth of the 2nd moment, $\langle |\Theta|^2 \rangle$, (which we define as a *weak clustering*) results in that binary collisions occur mainly between particles inside the clusters. The latter can be important in coagulation of droplets in atmospheric clouds whereby the collisions between droplets play a crucial role in a rain formation. The growth of the q-th moment, $\langle |\Theta|^q \rangle$, results in that q-particles collisions occur mainly between particles inside the cluster. The growth of the negative moments of particles number density (possibly associated with formation of voids and cellular structures) was discussed in [9] (see also [10,11]).

III. GROWTH OF THE 2ND MOMENT

In a previous section we estimated the growth rates of all moments $\langle |\Theta|^q \rangle$. Here we present the results of a rigorous analysis for the evolution of the two-point 2nd moment

$$\Phi(t, \boldsymbol{R}) \equiv \langle \Theta(t, \boldsymbol{r}) \Theta(t, \boldsymbol{r} + \boldsymbol{R}) \rangle \; . \tag{7}$$

In this analysis we used stochastic calculas (*e.g.,* Wiener path integral representation of the solution of the Cauchy problem for eq. (1), Feynman-Kac formula and Cameron-Martin-Girsanov theorem). We showed that a finite correlation time of a turbulent velocity plays a crucial role for the clustering instability. Notably, an equation for the second moment $\Phi(t, \boldsymbol{R})$ of the number density of inertial particles comprises spatial derivatives of high orders due to a non-local nature of turbulent transport of inertial particles in a random velocity field with a finite correlation time [8]. However, we found that at least for two models of a random velocity field [(i) the random velocity with Gaussian statistics of the integrals $\int_0^t \boldsymbol{v}(t', \boldsymbol{\xi})\, dt'$ and $\int_0^t b(t', \boldsymbol{\xi})\, dt'$; and (ii) the Gaussian velocity field with a small yet finite correlation time] the equation for $\Phi(t, \boldsymbol{R})$ is a second-order partial differential equation:

$$\begin{aligned} &\partial \Phi / \partial t = \hat{\mathcal{L}}\, \Phi(t, \boldsymbol{R}) \, , \\ &\hat{\mathcal{L}} = B(\boldsymbol{R}) + 2\boldsymbol{U}(\boldsymbol{R}) \cdot \boldsymbol{\nabla} + \hat{D}_{\alpha\beta}(\boldsymbol{R}) \nabla_\alpha \nabla_\beta \; . \end{aligned} \tag{8}$$

For the first model of the turbulent velocity field, *e.g.,* the coefficients in eq. (8) are given by:

$$B(\boldsymbol{R}) \approx 2 \int_0^\infty \langle b[0, \boldsymbol{\xi}(\boldsymbol{r}_1|0)] b[\tau, \boldsymbol{\xi}(\boldsymbol{r}_2|\tau)]\rangle \, d\tau \,, \tag{9}$$
$$\boldsymbol{U}(\boldsymbol{R}) \approx -2 \int_0^\infty \langle \boldsymbol{v}[0, \boldsymbol{\xi}(\boldsymbol{r}_1|0)] b[\tau, \boldsymbol{\xi}(\boldsymbol{r}_2|\tau)]\rangle \, d\tau \,,$$
$$\tilde{D}^{\rm T}_{\alpha\beta}(\boldsymbol{R}) \approx 2 \int_0^\infty \langle v_\alpha[0, \boldsymbol{\xi}(\boldsymbol{r}_1|0)] v_\beta[\tau, \boldsymbol{\xi}(\boldsymbol{r}_2|\tau)]\rangle \, d\tau \,,$$

where $\hat{D}_{\alpha\beta}(\boldsymbol{R}) = 2D\delta_{\alpha\beta} + D^{\rm T}_{\alpha\beta}(\boldsymbol{R})$ and $D^{\rm T}_{\alpha\beta}(\boldsymbol{R}) = \tilde{D}^{\rm T}_{\alpha\beta}(0) - \tilde{D}^{\rm T}_{\alpha\beta}(\boldsymbol{R})$ is the scale-dependent tensor of turbulent diffusion.

For the δ-correlated in time random Gaussian compressible velocity field the operator $\hat{\mathcal{L}}$ is replaced by $\hat{\mathcal{L}}_0$ in the equation for the second moment $\Phi(t, \boldsymbol{R})$, where

$$\hat{\mathcal{L}}_0 \equiv B_0(\boldsymbol{R}) + 2\boldsymbol{U}_0(\boldsymbol{R}) \cdot \boldsymbol{\nabla} + \hat{D}_{\alpha\beta}(\boldsymbol{R}) \nabla_\alpha \nabla_\beta \,, \tag{10}$$
$$B_0(\boldsymbol{R}) = \nabla_\alpha \nabla_\beta \hat{D}_{\alpha\beta}(\boldsymbol{R}) \,, \quad U_{0,\alpha}(\boldsymbol{R}) = \nabla_\beta \hat{D}_{\alpha\beta}(\boldsymbol{R}) \tag{11}$$

(for details see [8,12]). In the δ-correlated in time velocity field the second moment $\Phi(t, \boldsymbol{R})$ can only decay in spite of the compressibility of the velocity field. The reason is that the differential operator $\hat{\mathcal{L}}_0 \equiv \nabla_\alpha \nabla_\beta \hat{D}_{\alpha\beta}(\boldsymbol{R})$ is adjoint to the operator $\hat{\mathcal{L}}_0^\dagger \equiv \hat{D}_{\alpha\beta}(\boldsymbol{R}) \nabla_\alpha \nabla_\beta$ and their eigenvalues are equal. The damping rate for the equation

$$\partial \Phi / \partial t = \hat{\mathcal{L}}_0^\dagger \, \Phi(t, \boldsymbol{R}) \tag{12}$$

has been found in ref. [13] for a compressible isotropic homogeneous turbulence in a dissipative range:

$$\gamma_2 = - \frac{(3 - \sigma_{\rm T})^2}{6 \, \tau_\eta (1 + \sigma_{\rm T})(1 + 3\sigma_{\rm T})} \,, \tag{13}$$

where $\sigma_{\rm T}$ is the degree of compressibility of the tensor $D^{\rm T}_{\alpha\beta}(\boldsymbol{R})$. For the δ-correlated in time incompressible velocity field ($\sigma_{\rm T} = 0$) eq. (12) was derived in ref. [14]. Thus, for the model of turbulent advection with a delta correlated in time velocity field the clustering instability of the 2nd moment does not occur.

A general form of the turbulent diffusion tensor in a dissipative range is given by

$$D^{\rm T}_{\alpha\beta}(\boldsymbol{R}) = (C_1 R^2 \delta_{\alpha\beta} + C_2 R_\alpha R_\beta)/\tau_\eta \,, \tag{14}$$
$$C_1 = (4 + 2\sigma_{\rm T})/3\,(1 + \sigma_{\rm T}) \,, \; C_2 = (4\sigma_{\rm T} - 2)/3\,(1 + \sigma_{\rm T}).$$

The parameter $\sigma_{\rm T}$ is defined by analogy with eq. (4):

$$\sigma_{\rm T} \equiv \frac{\boldsymbol{\nabla} \cdot \boldsymbol{D}_{\rm T} \cdot \boldsymbol{\nabla}}{\boldsymbol{\nabla} \times \boldsymbol{D}_{\rm T} \times \boldsymbol{\nabla}} = \frac{\nabla_\alpha \nabla_\beta D^{\rm T}_{\alpha\beta}(\boldsymbol{R})}{\nabla_\alpha \nabla_\beta D^{\rm T}_{\alpha'\beta'}(\boldsymbol{R}) \epsilon_{\alpha\alpha'\gamma} \epsilon_{\beta\beta'\gamma}} \,, \tag{15}$$

where $\epsilon_{\alpha\beta\gamma}$ is the fully antisymmetric unit tensor. By definitions (4) and (15) $\sigma_{\rm T} = \sigma_v$ in the case of δ-correlated in time compressible velocity field. Equations (9) show that for a finite correlation time identities (11) are violated: $B(\boldsymbol{R}) \neq B_0(\boldsymbol{R})$, $\boldsymbol{U}(\boldsymbol{R}) \neq \boldsymbol{U}_0(\boldsymbol{R})$.

Let us study the clustering instability. Equation (8) in a nondimensional form reads:

$$\frac{\partial \Phi}{\partial \tilde{t}} = \frac{\Phi''}{m(r)} + \Big[\frac{1}{m(r)} + (U - C_2) r^2\Big] \frac{2\,\Phi'}{r} + B\Phi \,,$$
$$1/m(r) \equiv (C_1 + C_2) r^2 + 2/{\rm Sc} \,, \tag{16}$$

where $\boldsymbol{U} \equiv U\boldsymbol{R}$ and ${\rm Sc} = \nu/D$ is the Schmidt number. For inertial particles ${\rm Sc} \gg 1$. The nondimensional variables in eq. (16) are $r \equiv R/\eta$ and $\tilde{t} = t/\tau_\eta$, B and U are measured in the units τ_η^{-1}. In a *molecular diffusion region of scales* whereby $r \ll {\rm Sc}^{-1/2}$, all terms $\propto r^2$ (with C_1 , C_2 and U) may be neglected. Then the solution of eq. (16) is given by $\Phi(r) = (1 - \alpha r^2) \exp(\gamma_2 t)$, where $\alpha = {\rm Sc}(B - \gamma_2 \tau_\eta)/12$ and $B > \gamma_2 \tau_\eta$. In a *turbulent diffusion region of scales*, ${\rm Sc}^{-1/2} \ll r \ll 1$, the molecular diffusion term $\propto 1/{\rm Sc}$ is negligible. Thus, the solution of eq. (16) in this region is $\Phi(r) = A_1 r^{-\lambda} \exp(\gamma_2 t)$, where $\lambda = (C_1 - C_2 + 2U \pm iC_3)/2(C_1 + C_2)$, and $C_3^2 = 4(B - \gamma_2 \tau_\eta)(C_1 + C_2) - (C_1 - C_2 + 2U)^2$. Since the total number of particles is conserved in a closed volume, $\int_0^\infty r^2 \Phi(r) \, dr = 0$. This implies that $C_3^2 > 0$, and therefore λ is a complex number. In addition, the correlation function $\Phi(r)$ has global maximum at $r = 0$. This implies that $C_1 > C_2 - 2U$. The latter condition, *e.g.,* for very small U yields $\sigma_T \leq 3$. For $r \gg 1$ the solution for $\Phi(r)$ decays sharply with r. The growth rate γ_2 of the second moment of particles number density can be obtained by matching the correlation function $\Phi(r)$ and its first derivative $\Phi'(r)$ at the boundaries of the above regions, i.e., at the points $r = {\rm Sc}^{-1/2}$ and $r = 1$. The matching yields $C_3/2(C_1 + C_2) \approx 2\pi/\ln {\rm Sc}$. Thus,

$$\gamma_2 = \frac{1}{\tau_\eta (1 + 3\sigma_{\rm T})} \Big[\frac{200 \sigma_{\rm U} (\sigma_{\rm T} - \sigma_{\rm U})}{3(1 + \sigma_{\rm U})} - \frac{(3 - \sigma_{\rm T})^2}{6(1 + \sigma_{\rm T})} \tag{17}$$
$$- \frac{3\pi^2 (1 + 3\sigma_{\rm T})^2}{(1 + \sigma_{\rm T}) \ln^2 {\rm Sc}}\Big] + \frac{20(\sigma_{\rm B} - \sigma_{\rm U})}{\tau_\eta (1 + \sigma_{\rm B})(1 + \sigma_{\rm U})} \,,$$

where we introduced $\sigma_{\rm B}$ and $\sigma_{\rm U}$ defined by equations $B = 20\sigma_{\rm B}/(1 + \sigma_{\rm B})$ and $U = 20\sigma_{\rm U}/3(1 + \sigma_{\rm U})$. Note that the parameters $\sigma_{\rm B} \approx \sigma_{\rm U} \sim \sigma_v$. For the δ-correlated in time random compressible velocity field $\sigma_{\rm B} = \sigma_{\rm U} = \sigma_{\rm T} = \sigma_v$. Note that eq. (13) is written for ${\rm Sc} \to \infty$. Analysis of eq. (17) shows that the critical value of σ_v required for the clustering instability is $\sigma_{\rm cr} \approx (0.1 - 0.2)$. Notably, in the second model of a random velocity field (*i.e.,* the Gaussian velocity field with a small yet finite correlation time) the clustering instability occurs when $\sigma_v > 0.2$. Equation (6) also yields for the threshold of the instability of the 2nd moment (at $q_{\rm cr} = 2$) a similar value $\sigma_{\rm cr} \sim 1/6$.

IV. NONLINEAR EFFECTS

The compressibility of the turbulent velocity field with a finite correlation time can cause the exponential growth

of the moments of particles number density. This small-scale instability results in formation of strong inhomogeneities (clusters) in particles spatial distributions. The linear analysis does not allow to determine a mechanism of saturation of the clustering instability. As can be seen from eq. (17) molecular diffusion only depletes the growth rates of the clustering instability at the linear stage (contrary to the instability discussed in ref. [9]). The clustering instability is saturated by nonlinear effects.

Consider now a mechanism of the nonlinear saturation of the clustering instability using on the example of atmospheric turbulence with characteristic parameters: $\eta \sim 1$mm, $\tau_\eta \sim (0.1 - 0.01)$s. A momentum coupling of particles and turbulent fluid may be essential when the mass loading is not small: $m_{\rm p} n_{\rm cl} \sim \rho$. For this condition the kinetic energy of fluid $\rho \langle \boldsymbol{u}^2 \rangle$ is of the order of the particles energy $m_{\rm p} n_{\rm cl} \langle \boldsymbol{v}^2 \rangle$, where $|\boldsymbol{u}| \sim |\boldsymbol{v}|$. This yields:

$$n_{\rm cl} \sim a^{-3} (\rho / 3\rho_{\rm p}) \; . \tag{18}$$

For water droplets $\rho_{\rm p}/\rho \sim 10^3$. Thus, for $a = a_* \sim 30\mu$m we obtain $n_{\rm cl} \sim 10^4$ cm^{-3} and the total number of particles in the cluster of size η, $N_{\rm cl} \simeq \eta^3 n_{\rm cl} \sim 10$. These values may be considered as a lower estimate for coupling effects of particles on the flow.

An actual mechanism of the nonlinear saturation of the clustering instability is associated with particles collisions causing particles effective pressure which prevents further grows of concentration. Particles collisions play essential role when during the life-time of a cluster the total number of collisions is of the order of number of particles in the cluster. The rate of collisions $J \sim n_{\rm cl}/\tau_\eta$ can be estimated as $J \sim 4\pi a^2 n_{\rm cl}^2 |\boldsymbol{v}_{\rm rel}|$. The relative velocity $\boldsymbol{v}_{\rm rel}$ of colliding particles with different but comparable sizes can be estimated as $|\boldsymbol{v}_{\rm rel}| \sim \tau_{\rm p} |(\boldsymbol{u} \cdot \boldsymbol{\nabla}) \boldsymbol{u}| \sim \tau_{\rm p} u_\eta^2 / \eta$. Thus the collisions in clusters may be essential for

$$n_{\rm cl} \sim a^{-3} (\eta / a)(\rho / 3\rho_{\rm p}) \, , \quad \ell_{\rm s} \sim a (3a\rho_{\rm p} / \eta \rho)^{1/3} \, , \tag{19}$$

where $\ell_{\rm s}$ is a mean separation of particles in the cluster. For the above parameters ($a = 30\mu$m) $n_{\rm cl} \sim 3 \times 10^5$cm^{-3}, $\ell_{\rm s} \sim 5\,a \approx 150\mu$m and $N_{\rm cl} \sim 300$. Note that the mean number density of droplets in clouds $\bar{n}$ is about $10^2 - 10^3$cm^{-3}. Therefore the *clustering instability of droplets in the clouds increases their concentrations in the clusters by the orders of magnitude.*

In all our analysis we have neglected an effect of sedimentation of particles in gravity field which is essential for particles of the size $a > 100\mu$m. Taking $\ell_{\rm cl} \simeq \eta$ we assumed implicitly that $\tau_{\rm p} < \tau_\eta$. This is valid (for the atmospheric conditions) if $a \leq 60\mu$m. Otherwise the cluster size can be estimated as $\ell_{\rm cl} \simeq \eta (\tau_{\rm p} / \tau_\eta)^{3/2}$.

Our estimates support the suggestion that *the clustering instability serves as a preliminary stage for a coagulation of water droplets in clouds leading to a rain formation.*

V. SUMMARY

• We showed that the physical reason for the *clustering instability* in spatial distribution of concentration of particles in turbulent flows is a combined effect of the inertia of particles leading to a compressibility of the particle velocity field $\boldsymbol{v}(t, \boldsymbol{r})$ and finite velocity correlation time.

• The clustering instability may result in a *strong clustering* in which a finite fraction of particles are accumulated in the clusters and a *weak clustering* when a finite fraction of particle collisions occurs in the clusters.

• The crucial parameter for the clustering instability is a radius of the particles a. The instability criterion is $a > a_{\rm cr} \approx a_*$ for which $\langle (\mathrm{div}\, \boldsymbol{v})^2 \rangle = \langle |\mathrm{rot}\, \boldsymbol{v}|^2 \rangle$. For the droplets in the atmosphere $a_* \simeq 30\mu$m. The growth rate of the clustering instability $\gamma_{\rm cl} \sim \tau_\eta^{-1} (a/a_*)^4$, where τ_η is the turnover time in the viscous scales of turbulence.

• We introduced a new concept of compressibility of the turbulent diffusion tensor caused by a finite correlation time of an incompressible velocity field. For this model of velocity field, the field of Lagrangian trajectories is not divergence-free.

• We suggested a mechanism of saturation of the clustering instability - *particle collisions in the clusters*. An evaluated nonlinear level of the saturation of the droplets number density in clouds exceeds by the orders of magnitude their mean number density.

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